LATTICES RELATED TO EXTENSIONS OF PRESENTATIONS OF TRANSVERSAL MATROIDS

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ABSTRACT. For a presentation $\mathcal{A}$ of a transversal matroid M, we study the set $T_{\mathcal{A}}$ of single-element transversal extensions of M that have presentations that extend $\mathcal{A}$; we order these extensions by the weak order. We show that $T_{\mathcal{A}}$ is a distributive lattice, and that each finite distributive lattice is isomorphic to $T_{\mathcal{A}}$ for some presentation $\mathcal{A}$ of some transversal matroid M. We show that $T_{\mathcal{A}} \cap T_{\mathcal{B}}$, for any two presentations $\mathcal{A}$ and $\mathcal{B}$ of M, is a sublattice of both $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$. We prove sharp upper bounds on $|T_{\mathcal{A}}|$ for presentations $\mathcal{A}$ of rank less than $r(M)$ in the order on presentations; we also give a sharp upper bound on $|T_{\mathcal{A}} \cap T_{\mathcal{B}}|$. The main tool we introduce to study $T_{\mathcal{A}}$ is the lattice $L_{\mathcal{A}}$ of closed sets of a certain closure operator on the lattice of subsets of $\{1, 2, \dots, r(M)\}$.

1. INTRODUCTION

We continue the investigation, begun in [4], of the extent to which a presentation $\mathcal{A}$ of a transversal matroid M limits the single-element transversal extensions of M that can be obtained by extending $\mathcal{A}$. The following analogy may help orient readers. A matrix A, over a field $\mathbb{F}$, that represents a matroid M may contain extraneous information; this can limit which $\mathbb{F}$-representable single-element extensions of M can be represented by extending (i.e., adjoining another column to) A. For instance, for the rank-3 uniform matroid $U_{3,6}$, partition $E(U_{3,6})$ into three 2-point lines, L_1, L_2, and L_3. Let A be a 3×6 matrix, over $\mathbb{F}$, that represents $U_{3,6}$. The line L_i is represented by a pair of columns of A, which span a 2-dimensional subspace V_i of $\mathbb{F}^3$. While $V_i \cap V_j$, for $\{i, j\} \subset \{1, 2, 3\}$, has dimension 1 (since the corresponding lines of $U_{3,6}$ are coplanar), the intersection $V_1 \cap V_2 \cap V_3$ can, in general, have dimension either 0 or 1: this dimension is extraneous. If $\dim(V_1 \cap V_2 \cap V_3)$ is 1, then no extension of A represents the extension of M that has an element on, say, L_1 and L_2 but not L_3; otherwise, no extension of A represents the extension of M that has a non-loop on all three lines. (The underlying problem is the lack of unique representability, which is a major complicating factor for research on representable matroids. See Oxley [12, Section 14.6].) In this paper, we consider such problems, but for transversal matroids in place of $\mathbb{F}$-representable matroids, and presentations in place of matrix representations.

A transversal matroid can be given by a presentation, which is a sequence of sets whose partial transversals are the independent sets. In [4], we introduced the ordered set $T_{\mathcal{A}}$ of transversal single-element extensions of M that have presentations that extend $\mathcal{A}$ (i.e., the new element is adjoined to some of the sets in $\mathcal{A}$), where we order extensions by the weak order. In Section 3, we introduce a new tool for studying $T_{\mathcal{A}}$: given a presentation $\mathcal{A}$ of a transversal matroid M with the number, $|\mathcal{A}|$, of terms in the sequence $\mathcal{A}$ being the rank, r, of M, we define a closure operator on the lattice $2^{[r]}$ of subsets of the set $[r] = \{1, 2, \dots, r\}$, and we show that the resulting lattice $L_{\mathcal{A}}$ of closed sets is a (necessarily

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distributive) sublattice of $2^{[r]}$ that is isomorphic to $T_{\mathcal{A}}$. While they are isomorphic, $L_{\mathcal{A}}$ is often simpler to work with than is $T_{\mathcal{A}}$. We prove some basic properties of the lattice $L_{\mathcal{A}}$, give several descriptions of its elements, show that every distributive lattice is isomorphic to $L_{\mathcal{A}}$, and so to $T_{\mathcal{A}}$, for a suitable choice of M and $\mathcal{A}$, and we interpret the join- and meet-irreducible elements of $L_{\mathcal{A}}$. We show that if $\mathcal{A}$ and $\mathcal{B}$ are both presentations of M, then $T_{\mathcal{A}} \cap T_{\mathcal{B}}$ is a sublattice of $T_{\mathcal{A}}$ and of $T_{\mathcal{B}}$. In [4], we showed that $|T_{\mathcal{A}}| = 2^r$ if and only if the presentation $\mathcal{A}$ of M is minimal in the natural order on the presentations of M; using $L_{\mathcal{A}}$, in Section 4 we prove upper bounds on $|T_{\mathcal{A}}|$ for the next r lowest ranks in this order. We also show that $|T_{\mathcal{A}} \cap T_{\mathcal{B}}| \leq \frac{3}{4} \cdot 2^r$ whenever presentations $\mathcal{A}$ and $\mathcal{B}$ of M differ by more than just the order of the sets.

The relevant background is recalled in the next section. See Brualdi [5] for more about transversal matroids, and Oxley [12] for other matroid background.

2. Background

A *set system* $\mathcal{A} = (A_i : i \in [r])$ on a set E is a sequence of subsets of E. A *partial transversal* of $\mathcal{A}$ is a subset X of E for which there is an injection $\phi : X \to [r]$ with $e \in A_{\phi(e)}$ for all $e \in X$; such an injection is an $\mathcal{A}$*-matching of* X *into* $[r]$. Edmonds and Fulkerson [9] showed that the partial transversals of $\mathcal{A}$ are the independent sets of a matroid on E; we say that $\mathcal{A}$ is a *presentation* of this *transversal matroid* $M[\mathcal{A}]$.

The first lemma is an easy observation.

Lemma 2.1. *Let M be $M[\mathcal{A}]$ with $\mathcal{A} = (A_i : i \in [r])$. For any subset X of $E(M)$, the restriction $M|X$ is transversal and $(A_i \cap X : i \in [r])$ is a presentation of $M|X$.*

We focus on presentations $(A_i : i \in [r])$ of M that are of the type guaranteed by the first part of Lemma 2.2, that is, $r = r(M)$; the second part of the lemma explains why other presentations are not substantially different.

Lemma 2.2. *Each transversal matroid M has a presentation $\mathcal{A}$ with $|\mathcal{A}| = r(M)$. If M has no coloops, then all presentations of M have exactly $r(M)$ nonempty sets (counting multiplicity).*

Given a presentation $\mathcal{A} = (A_i : i \in [r])$ of a transversal matroid M and a subset X of $E(M)$, the $\mathcal{A}$*-support*, $s_{\mathcal{A}}(X)$, *of* X is

$$s_{\mathcal{A}}(X) = \{i : X \cap A_i \neq \emptyset\}.$$

A *cyclic set* in a matroid M is a (possibly empty) union of circuits; thus, $X \subseteq E(M)$ is cyclic if and only if $M|X$ has no coloops. Lemmas 2.1 and 2.2 give the next result.

Corollary 2.3. *If X is a cyclic set of $M[\mathcal{A}]$, then $|s_{\mathcal{A}}(X)| = r(X)$.*

By Hall's theorem [1, Theorem VIII.8.20], a subset Y of $E(M)$ is independent in M if and only if $|s_{\mathcal{A}}(Z)| \geq |Z|$ for all subsets Z of Y. One can prove the next lemma from this.

Lemma 2.4. *Let $\mathcal{A}$ be a presentation of M.*

(1) *For any circuit C of M and element $e \in C$, we have*

$$|s_{\mathcal{A}}(C)| = |s_{\mathcal{A}}(C - \{e\})| = r(C) = |C| - 1,$$

so $s_{\mathcal{A}}(C) = s_{\mathcal{A}}(C - \{e\})$.

(2) *If $X \subseteq E(M)$ with $|s_{\mathcal{A}}(X)| = r(X)$, then its closure,* $\mathrm{cl}(X)$, *is*

$$\mathrm{cl}(X) = \{e : s_{\mathcal{A}}(e) \subseteq s_{\mathcal{A}}(X)\}.$$

Extending a presentation $\mathcal{A} = (A_i : i \in [r])$ of a transversal matroid M consists of adjoining an element x that is not in $E(M)$ to some of the sets in $\mathcal{A}$. More precisely, for an element $x \notin E(M)$ and a subset I of $[r]$, we let $\mathcal{A}^I$ be $(A_i^I : i \in [r])$ where

$$A_i^I = \begin{cases} A_i \cup \{x\}, & \text{if } i \in I, \\ A_i, & \text{otherwise.} \end{cases}$$

The matroid $M[\mathcal{A}^I]$ on the set $E(M) \cup \{x\}$ is a rank-preserving single-element extension of M. (This is the only type of extension we consider, so below we omit the adjectives "rank-preserving" and "single-element".) Throughout this paper, we reserve x for the element by which we extend a matroid.

We will use principal extensions of matroids, which we now recall. For any matroid M (not necessarily transversal), a subset Y of $E(M)$, and an element x that is not in $E(M)$, the *principal extension* $M +_Y x$ of M is the matroid on $E(M) \cup \{x\}$ with the rank function r' where, for $Z \subseteq E(M)$, we have $r'(Z) = r_M(Z)$ and

$$r'(Z \cup \{x\}) = \begin{cases} r_M(Z), & \text{if } Y \subseteq \mathrm{cl}_M(Z), \\ r_M(Z) + 1, & \text{otherwise.} \end{cases}$$

Thus, $M +_Y x = M +_{Y'} x$ whenever $\mathrm{cl}_M(Y) = \mathrm{cl}_M(Y')$. Geometrically, $M +_Y x$ is formed by putting x freely in the flat $\mathrm{cl}_M(Y)$. A routine argument using matchings and part (2) of Lemma 2.4 yields the following result.

Lemma 2.5. *Let $\mathcal{A}$ be a presentation of a transversal matroid M. If Y is a subset of $E(M)$ with $|s_{\mathcal{A}}(Y)| = r(Y)$, then $M[\mathcal{A}^{s_{\mathcal{A}}(Y)}]$ is the principal extension $M +_Y x$, and, relative to containment, the least cyclic flat of $M[\mathcal{A}^{s_{\mathcal{A}}(Y)}]$ that contains x is $\mathrm{cl}_M(Y) \cup \{x\}$.*

A transversal matroid typically has many presentations, and there is a natural order on them. A mild variant of the customary order on presentations best meets our needs. For presentations $\mathcal{A} = (A_i : i \in [r])$ and $\mathcal{B} = (B_i : i \in [r])$ of M, we set $\mathcal{A} \preceq \mathcal{B}$ if $A_i \subseteq B_i$ for all $i \in [r]$. We write $\mathcal{A} \prec \mathcal{B}$ if, in addition, at least one of these inclusions is strict. We say that $\mathcal{B}$ *covers* $\mathcal{A}$, and we write $\mathcal{A} \prec\!\cdot\, \mathcal{B}$, if $\mathcal{A} \prec \mathcal{B}$ and there is no presentation $\mathcal{C}$ of M with $\mathcal{A} \prec \mathcal{C} \prec \mathcal{B}$. (The customary order identifies $(A_i : i \in [r])$ and $(A_{\tau(i)} : i \in [r])$ for any permutation τ of $[r]$, and so sets $\mathcal{A} \leq \mathcal{B}$ if, up to re-indexing, $A_i \subseteq B_i$ for all $i \in [r]$.)

Mason [11] showed that if $(A_i : i \in [r])$ and $(B_i : i \in [r])$ are maximal presentations of the same transversal matroid, then there is a permutation τ of $[r]$ with $A_{\tau(i)} = B_i$ for all $i \in [r]$. (Minimal presentations, in contrast, are often more varied.) The next lemma, which is due to Bondy and Welsh [2] and plays important roles in this paper, gives a constructive way to find the maximal presentations of a transversal matroid.

Lemma 2.6. *Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of M. Let i be in $[r]$ and e in $E(M) - A_i$. The following statements are equivalent:*

(1) *the set system obtained from $\mathcal{A}$ by replacing A_i by $A_i \cup \{e\}$ is also a presentation of M, and*
(2) *e is a coloop of the deletion $M \backslash A_i$.*

A routine argument shows that the complement $E(M) - A_i$ of any set A_i in $\mathcal{A}$ is a flat of $M[\mathcal{A}]$. By Lemma 2.6, the complement of each set in a maximal presentation of M is a cyclic flat of M. Bondy and Welsh [2] and Las Vergnas [10] proved the next result about the sets in minimal presentations.

Lemma 2.7. *A presentation $(C_i : i \in [r])$ of M is minimal if and only if each set C_i is a cocircuit of M, that is, $E(M) - C_i$ is a hyperplane of M.*

Thus, $(C_i : i \in [r])$ is minimal if and only if $r(M\backslash C_i) = r - 1$ for all $i \in [r]$. The next result, by Brualdi and Dinolt [6], follows from the last two lemmas.

Lemma 2.8. *If $\mathcal{A} = (A_i : i \in [r])$ is a presentation of M and $\mathcal{C} = (C_i : i \in [r])$ is a minimal presentation of M with $\mathcal{C} \preceq \mathcal{A}$, then*

$$|A_i - C_i| = r(M\backslash C_i) - r(M\backslash A_i) = r - 1 - r(M\backslash A_i).$$

Corollary 2.9. *The ordered set of presentations of a rank-r transversal matroid M is ranked; the rank of a presentation $(A_i : i \in [r])$ is*

$$r(r-1) - \sum_{i=1}^{r} r(M\backslash A_i).$$

This corollary applies to both the order we focus on, $\mathcal{A} \preceq \mathcal{B}$, and the more customary order, $\mathcal{A} \leq \mathcal{B}$; the rank of a presentation is the same in both orders.

The *weak order* $\leq_w$ on matroids on the same set E is defined as follows: $M \leq_w N$ if $r_M(X) \leq r_N(X)$ for all subsets X of E; equivalently, every independent set of M is independent in N. This captures the idea that N is freer than M. The next two lemmas are simple but useful observations.

Lemma 2.10. *Let $M = M[(A_i : i \in [r])]$ and $N = M[(B_i : i \in [r])]$, where M and N are defined on the same set. If $A_i \subseteq B_i$ for all $i \in [r]$, then $M \leq_w N$.*

Lemma 2.11. *Assume that $M \leq_w N$ and $M\backslash e = N\backslash e$. If e is a coloop of M, then e is a coloop of N, and so $M = N$.*

Lastly, we recall how to think of transversal matroids geometrically and to give affine representations of those of low rank, as in Figures 1 and 2. A set system $\mathcal{A} = (A_i : i \in [r])$ on E can be encoded by a 0-1 matrix with r rows whose columns are indexed by the elements of E in which the i, e entry is 1 if and only if $e \in A_i$. If we replace the 1s in this matrix by distinct variables, say over $\mathbb{R}$, then it follows from the permutation expansion of determinants that the linearly independent columns are precisely the partial transversals of $\mathcal{A}$, so this is a matrix representation of $M[\mathcal{A}]$. One can in turn replace the variables by non-negative real numbers and preserve which square submatrices have nonzero determinants; one can also scale the columns so that the sum of the entries in each nonzero column is 1. In this way, each non-loop of M is represented by a point in the convex hull of the standard basis vectors. This yields the following geometric picture: label the vertices of a simplex $1, 2, \ldots, r$ and think of associating A_i to the i-th vertex, then place each point e of E freely (relative to the other points) in the face of the simplex spanned by $s_{\mathcal{A}}(e)$.

3. A closure operator and two isomorphic distributive lattices

Let $\mathcal{A}$ be a presentation of M. In [4], we introduced the ordered set $T_{\mathcal{A}}$ of transversal extensions of M that have presentations that extend $\mathcal{A}$, ordering $T_{\mathcal{A}}$ by the weak order. As the results in this paper demonstrate, the lattice $L_{\mathcal{A}}$ of subsets of $[r(M)]$ that we define in this section and show to be isomorphic to $T_{\mathcal{A}}$ is very useful for studying $T_{\mathcal{A}}$.

Recall that we consider only single-element rank-preserving extensions. Also, x always denotes the element by which we extend a matroid.

3.1. **The lattice $L_{\mathcal{A}}$.** The first lattice we discuss is the lattice of closed sets for a closure operator that we introduce below, so we first recall closure operators (see, e.g., [1, p. 49]). A *closure operator* on a set S is a map $\sigma : 2^S \to 2^S$ for which

(1) $X \subseteq \sigma(X)$ for all $X \subseteq S$,

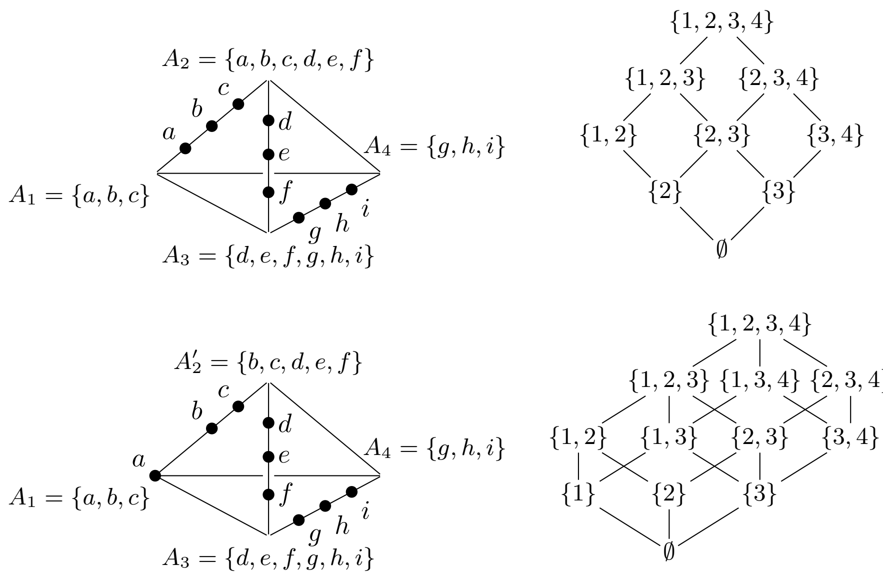

FIGURE 1. Two presentations $\mathcal{A}$ of a transversal matroid M, along with the associated lattices $L_{\mathcal{A}}$.

(2) if $X \subseteq Y \subseteq S$, then $\sigma(X) \subseteq \sigma(Y)$, and
(3) $\sigma(\sigma(X)) = \sigma(X)$ for all $X \subseteq S$.

Given a closure operator $\sigma : 2^S \to 2^S$, a *σ-closed set* is a subset X of S with $\sigma(X) = X$. The set of σ-closed sets, ordered by containment, is a lattice; join and meet are given by $X \vee Y = \sigma(X \cup Y)$ and $X \wedge Y = X \cap Y$. By property (1), the set S is σ-closed.

Let $\mathcal{A}$ be a presentation of a rank-r transversal matroid M. By Lemma 2.6, for each subset I of $[r]$, there is a greatest subset K of $[r]$, relative to containment, for which $M[\mathcal{A}^I] = M[\mathcal{A}^K]$, namely

$$K = I \cup \{k \in [r] - I \,:\, x \text{ is a coloop of } (M[\mathcal{A}^I])\backslash A_k\};$$

define a map $\sigma_{\mathcal{A}} : 2^{[r]} \to 2^{[r]}$ by setting $\sigma_{\mathcal{A}}(I) = K$. We next show that $\sigma_{\mathcal{A}}$ is a closure operator. We use $L_{\mathcal{A}}$ to denote the lattice of $\sigma_{\mathcal{A}}$-closed sets. See Figure 1 for examples.

Theorem 3.1. *For any presentation $\mathcal{A} = (A_i \,:\, i \in [r])$ of a transversal matroid M, the map $\sigma_{\mathcal{A}}$ defined above is a closure operator on $[r]$. The join in the lattice $L_{\mathcal{A}}$ of $\sigma_{\mathcal{A}}$-closed sets is given by $I \vee J = I \cup J$, so $L_{\mathcal{A}}$ is distributive. Both $\emptyset$ and $[r]$ are in $L_{\mathcal{A}}$.*

Proof. Properties (1) and (3) of closure operators clearly hold. For property (2), assume $I \subseteq J \subseteq [r]$ and $h \in \sigma_{\mathcal{A}}(I) - I$, so x is a coloop of $M[\mathcal{A}^I]\backslash A_h$. Lemma 2.10 gives $M[\mathcal{A}^I]\backslash A_h \leq_w M[\mathcal{A}^J]\backslash A_h$, so x is a coloop of $M[\mathcal{A}^J]\backslash A_h$ by Lemma 2.11, so $h \in \sigma(J)$, as needed.

Let I and J be in $L_{\mathcal{A}}$. Their meet, $I \wedge J$, is $I \cap J$ since, as noted above, this holds for any closure operator. We claim that $I \vee J = I \cup J$. (The fact that $L_{\mathcal{A}}$ is distributive then follows since union and intersection distribute over each other.) Since I and J are in $L_{\mathcal{A}}$,

(1) if $h \in [r] - I$, then x is not a coloop of $M[\mathcal{A}^I]\backslash A_h$, and
(2) if $h \in [r] - J$, then x is not a coloop of $M[\mathcal{A}^J]\backslash A_h$.

Note that the following two statements are equivalent: (i) $I \vee J = I \cup J$ and (ii) $I \cup J$ is $\sigma_{\mathcal{A}}$-closed. To prove statement (ii), let h be in $[r] - (I \cup J)$ and let Z be a basis of $M \backslash A_h$. If x were a coloop of $M[\mathcal{A}^{I \cup J}] \backslash A_h$, then there would be an $\mathcal{A}^{I \cup J}$-matching $\phi : Z \cup \{x\} \to [r]$. Either $\phi(x) \in I$ or $\phi(x) \in J$; if $\phi(x) \in I$, then ϕ shows that $Z \cup \{x\}$ is independent in $M[\mathcal{A}^I] \backslash A_h$, contrary to item (1) above; similarly, $\phi(x) \in J$ contradicts item (2). Thus, as needed, x is not a coloop of $M[\mathcal{A}^{I \cup J}] \backslash A_h$.

Note that $\emptyset$ is in $L_{\mathcal{A}}$ since x is a loop of $M[\mathcal{A}^I]$ if and only if $I = \emptyset$. □

We now show how the order on presentations relates to the lattices of closed sets.

Theorem 3.2. *For two presentations $\mathcal{A} = (A_i : i \in [r])$ and $\mathcal{B} = (B_i : i \in [r])$ of M, if $\mathcal{A} \preceq \mathcal{B}$, then $L_{\mathcal{B}}$ is a sublattice of $L_{\mathcal{A}}$ and $M[\mathcal{A}^I] = M[\mathcal{B}^I]$ for all $I \in L_{\mathcal{B}}$.*

Proof. Fix I in $L_{\mathcal{B}}$. Set $M_{\mathcal{B}} = M[\mathcal{B}^I]$ and $M_{\mathcal{A}} = M[\mathcal{A}^I]$. For $i \in [r] - I$, the element x is not a coloop of $M_{\mathcal{B}} \backslash B_i$ since $I \in L_{\mathcal{B}}$. Now $M_{\mathcal{A}} \backslash B_i \leq_w M_{\mathcal{B}} \backslash B_i$, so x is not a coloop of $M_{\mathcal{A}} \backslash B_i$ by Lemma 2.11, so x is not a coloop of $M_{\mathcal{A}} \backslash A_i$. Thus, $I \in L_{\mathcal{A}}$, so $L_{\mathcal{B}}$ is a sublattice of $L_{\mathcal{A}}$. Lemma 2.6 and the following two claims give $M_{\mathcal{A}} = M_{\mathcal{B}}$:

(1) for each $i \in I$, each element of $(B_i \cup \{x\}) - (A_i \cup \{x\})$ (that is, $B_i - A_i$) is a coloop of $M_{\mathcal{A}} \backslash (A_i \cup \{x\})$ (that is, $M \backslash A_i$), and
(2) for each $i \in [r] - I$, each element of $B_i - A_i$ is a coloop of $M_{\mathcal{A}} \backslash A_i$.

By the hypothesis and Lemma 2.6, for all $i \in [r]$, each element of $B_i - A_i$ is a coloop of $M \backslash A_i$, so claim (1) holds. For claim (2), fix $i \in [r] - I$ and $y \in B_i - A_i$. As shown above, x is not a coloop of $M_{\mathcal{A}} \backslash B_i$; let C be a circuit of $M_{\mathcal{A}} \backslash B_i$ with $x \in C$. Thus, $y \notin C$. Assume, contrary to claim (2), that some circuit C' of $M_{\mathcal{A}} \backslash A_i$ contains y. Now $x \in C'$ since y is coloop of $M \backslash A_i$. By strong circuit elimination, applied in $M_{\mathcal{A}} \backslash A_i$, some circuit $C'' \subseteq (C \cup C') - \{x\}$ contains y; however C'' is a circuit of $M \backslash A_i$, which contradicts y being a coloop of $M \backslash A_i$. Thus, claim (2) holds. □

The corollary below is a theorem from [4].

Corollary 3.3. *For each transversal extension M' of M, there is a minimal presentation of M that can be extended to a presentation of M'.*

3.2. **The lattice $T_{\mathcal{A}}$.** The lattice $T_{\mathcal{A}}$ consists of the set $\{M[\mathcal{A}^I] : I \in L_{\mathcal{A}}\}$ of transversal extensions of M that have presentations that extend $\mathcal{A}$, which we order by the weak order. The next result relates $T_{\mathcal{A}}$ and $L_{\mathcal{A}}$.

Theorem 3.4. *Let $\mathcal{A}$ be a presentation of M. For any sets I and J in $L_{\mathcal{A}}$, we have $M[\mathcal{A}^I] \leq_w M[\mathcal{A}^J]$ if and only if $I \subseteq J$. Thus, the bijection $I \mapsto M[\mathcal{A}^I]$ from $L_{\mathcal{A}}$ onto $T_{\mathcal{A}}$ is a lattice isomorphism, so $T_{\mathcal{A}}$ is a distributive lattice.*

Proof. Assume that $M[\mathcal{A}^I] \leq_w M[\mathcal{A}^J]$. Any $\mathcal{A}^{I \cup J}$-matching ϕ of an independent set X of $M[\mathcal{A}^{I \cup J}]$ with $x \in X$ has $\phi(x)$ in either I or J, so X is independent in one of $M[\mathcal{A}^I]$ and $M[\mathcal{A}^J]$, and so, by the assumption, in $M[\mathcal{A}^J]$. Thus, $M[\mathcal{A}^{I \cup J}] \leq_w M[\mathcal{A}^J]$. The equality $M[\mathcal{A}^J] = M[\mathcal{A}^{I \cup J}]$ now follows by Lemma 2.10; thus, $J = I \cup J$ since J and $I \cup J$ are $\sigma_{\mathcal{A}}$-closed, so $I \subseteq J$. The other implication follows from Lemma 2.10. □

Corollary 3.5. *For presentations $\mathcal{A}$ and $\mathcal{B}$ of M, if $\mathcal{A} \preceq \mathcal{B}$, then $T_{\mathcal{B}}$ is a sublattice of $T_{\mathcal{A}}$.*

The converse of the corollary fails even under the more common order on presentations as we now show.

EXAMPLE 1. Consider the uniform matroid $U_{3,4}$ on $\{a, b, c, d\}$ and its presentations

$$\mathcal{A} = (\{a, b, d\}, \{a, c, d\}, \{b, c, d\}) \quad \text{and} \quad \mathcal{B} = (\{a, b, c\}, \{a, b, d\}, \{a, c, d\}).$$

It is easy to check that both $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ consist of just the extension by a loop, $U_{3,4} \oplus U_{0,0}$, and the free extension, $U_{3,5}$. Thus, $T_{\mathcal{A}} = T_{\mathcal{B}} = T_{\mathcal{C}}$, where $\mathcal{C}$ is a maximal presentation of $U_{3,4}$, that is, $\mathcal{C} = (\{a,b,c,d\}, \{a,b,c,d\}, \{a,b,c,d\})$.

From the next result, which is a reformulation of [4, Theorem 3.1], we see that we cannot recover the presentation $\mathcal{A}$ from $L_{\mathcal{A}}$.

Theorem 3.6. *A presentation $\mathcal{A} = (A_i : i \in [r])$ of a transversal matroid M is minimal if and only if $L_{\mathcal{A}} = 2^{[r]}$, that is, $|T_{\mathcal{A}}| = 2^r$.*

Proof. If $\mathcal{A}$ is not minimal, then $r(M\backslash A_i) < r-1$ for some $i \in [r]$; thus, x is a coloop of $M[\mathcal{A}^{[r]-\{i\}}]\backslash A_i$, so $[r]-\{i\} \notin L_{\mathcal{A}}$. If $\mathcal{A}$ is minimal, then x is not a coloop of $M[\mathcal{A}^{\{i\}}]\backslash A_j$ for distinct $i,j \in [r]$ since $r(M\backslash A_j) = r-1$; thus, $\{i\} \in L_{\mathcal{A}}$, so closure under unions gives $L_{\mathcal{A}} = 2^{[r]}$. □

As Example 1 shows, we cannot always reconstruct the sets in $\mathcal{A}$ from $T_{\mathcal{A}}$; however, in some cases we can. For the matroid in Figure 1, one can check that the sets in each of its presentations $\mathcal{A}$ can be reconstructed from $T_{\mathcal{A}}$. Also, as we now show, for any transversal matroid M, the sets in each minimal presentation $\mathcal{A}$ of M can be reconstructed from $T_{\mathcal{A}}$. By Theorem 3.6, from $T_{\mathcal{A}}$, we know whether $\mathcal{A}$ is minimal. If $\mathcal{A}$ is minimal, remove the free extension, $M[\mathcal{A}^{[r]}]$, from $T_{\mathcal{A}}$; under the weak order, the maximal extensions left are $M[\mathcal{A}^I]$ with $I = [r]-\{i\}$ for $i \in [r]$; such an extension $M[\mathcal{A}^I]$ is, by Lemma 2.5, the principal extension $M +_{H_i} x$ of M, where H_i is the hyperplane of M that is the complement, $E(M) - A_i$, of the cocircuit A_i; also, $H_i \cup \{x\}$ is the unique cyclic hyperplane that contains x; thus, we can reconstruct each set A_i in $\mathcal{A}$.

3.3. **The sets in $L_{\mathcal{A}}$.** The results in this section, other than Corollary 3.8, are used heavily in Section 4. We start with several characterizations of the sets in $L_{\mathcal{A}}$.

Theorem 3.7. *For a presentation $\mathcal{A}$ of a transversal matroid M, the sets in $L_{\mathcal{A}}$ are*

(1) *the sets $s_{\mathcal{A}}(X)$, where X is an independent set of M and $|X| = |s_{\mathcal{A}}(X)|$, and*
(2) *all intersections of such sets.*

In particular, for $I \in L_{\mathcal{A}}$, if $\mathcal{C}_x$ is the set of all circuits of $M[\mathcal{A}^I]$ that contain x, then

$$I = \bigcap_{C \in \mathcal{C}_x} s_{\mathcal{A}}(C - \{x\}). \tag{3.1}$$

Item (1) *could be replaced by:* (1′) *the sets $s_{\mathcal{A}}(Y)$ where $r(Y) = |s_{\mathcal{A}}(Y)|$.*

Proof. Set $r = r(M)$. First assume that X satisfies condition (1). Set $I = s_{\mathcal{A}}(X)$. Thus, $X \cup \{x\}$ is dependent in $M[\mathcal{A}^I]$ but independent in $M[\mathcal{A}^{I\cup\{h\}}]$ for any $h \in [r]-I$, so I is in $L_{\mathcal{A}}$. Since $L_{\mathcal{A}}$ is closed under intersection, all sets identified above are in $L_{\mathcal{A}}$.

Fix I in $L_{\mathcal{A}}$ and let $\mathcal{C}_x$ be as defined above. Let X be $C - \{x\}$ for some $C \in \mathcal{C}_x$, so X is independent in M. Now $s_{\mathcal{A}}(X) = s_{\mathcal{A}^I}(X)$, and Lemma 2.4 gives $|s_{\mathcal{A}^I}(X)| = |X|$, so $|X| = |s_{\mathcal{A}}(X)|$. Also, $I = s_{\mathcal{A}^I}(x) \subseteq s_{\mathcal{A}^I}(C) = s_{\mathcal{A}}(X)$, so to prove equation (3.1) and show that all sets in $L_{\mathcal{A}}$ are given by items (1) and (2), it suffices to show that for each h in $[r]-I$, there is some $C_h \in \mathcal{C}_x$ with $h \notin s_{\mathcal{A}}(C_h - \{x\})$. Now $M[\mathcal{A}^I] \lneq_w M[\mathcal{A}^{I\cup\{h\}}]$, so some circuit, say C_h, of $M[\mathcal{A}^I]$ is independent in $M[\mathcal{A}^{I\cup\{h\}}]$. Thus, $C_h \in \mathcal{C}_x$ and

$$|s_{\mathcal{A}^{I\cup\{h\}}}(C_h)| \geq |C_h| > |s_{\mathcal{A}^I}(C_h)|,$$

so $h \notin s_{\mathcal{A}^I}(C_h)$, so $h \notin s_{\mathcal{A}}(C_h - \{x\})$, as needed.

Item (1′) can replace item (1) since, by Lemma 2.4, $r(Y) = |s_{\mathcal{A}}(Y)|$ for a set Y if and only if $|X| = |s_{\mathcal{A}}(X)|$ for some (equivalently, every) basis X of $M|Y$. □

By Lemma 2.5, in terms of $T_{\mathcal{A}}$, the extension that corresponds to a set $s_{\mathcal{A}}(X)$ in item (1) of Theorem 3.7 is the principal extension, $M +_X e$.

Corollary 3.8. *Let $\mathcal{A} = (A_i : i \in [r])$ be a presentation of M. If $F_1, F_2, \dots, F_k$ are cyclic flats of M, then $\bigcap_{i=1}^k s_{\mathcal{A}}(F_i) \in L_{\mathcal{A}}$. If $\mathcal{A}$ is a maximal presentation of M, then $L_{\mathcal{A}}$ consists of all such sets (which include $\emptyset$), along with $[r]$.*

Proof. The first assertion follows from Theorem 3.7 since cyclic flats satisfy condition (1′). Now let $\mathcal{A}$ be maximal. By Theorem 3.7, it suffices to show that if X is an independent set of M with $|X| = |s_{\mathcal{A}}(X)|$, then $s_{\mathcal{A}}(X)$ is the intersection of the $\mathcal{A}$-supports of some set of cyclic flats. Since $\mathcal{A}$ is maximal, each flat $E(M) - A_h$ of M, with $h \in [r]$, is cyclic by Lemma 2.6. If $h \in [r] - s_{\mathcal{A}}(X)$, then $X \subseteq E(M) - A_h$, so $s_{\mathcal{A}}(X) \subseteq s_{\mathcal{A}}\big(E(M) - A_h\big)$; also $h \notin s_{\mathcal{A}}\big(E(M) - A_h\big)$. Thus, as needed,

$$s_{\mathcal{A}}(X) = \bigcap_{h \in [r] - s_{\mathcal{A}}(X)} s_{\mathcal{A}}\big(E(M) - A_h\big). \qquad \square$$

The next result identifies some closed sets in terms of known closed sets and supports.

Corollary 3.9. *Let $\mathcal{A}$ be a presentation of M. Fix $F \subseteq E(M)$ and $J \in L_{\mathcal{A}}$, and set $H = s_{\mathcal{A}}(F) - J$. If $|H| \le |F|$ and $H \subseteq s_{\mathcal{A}}(e)$ for all $e \in F$, then $J \cup s_{\mathcal{A}}(F) \in L_{\mathcal{A}}$. In particular, if $s_{\mathcal{A}}(e) - \{h\} \in L_{\mathcal{A}}$ for some $e \in E(M)$ and $h \in s_{\mathcal{A}}(e)$, then $s_{\mathcal{A}}(e) \in L_{\mathcal{A}}$.*

Proof. Since $J \in L_{\mathcal{A}}$, there is a set $\mathcal{J}$ of subsets X of $E(M)$, all satisfying condition (1) of Theorem 3.7, with $J = \bigcap_{X \in \mathcal{J}} s_{\mathcal{A}}(X)$. For each set $X \in \mathcal{J}$, form a new set X' by adjoining any $|s_{\mathcal{A}}(F) - s_{\mathcal{A}}(X)|$ elements of F to X. Note that X' is independent: match elements in $X' - X$ to $s_{\mathcal{A}}(F) - s_{\mathcal{A}}(X)$. Now $s_{\mathcal{A}}(X') = s_{\mathcal{A}}(X \cup F)$ and

$$J \cup s_{\mathcal{A}}(F) = \bigcap_{X' : X \in \mathcal{J}} s_{\mathcal{A}}(X').$$

Also, $|X'| = |s_{\mathcal{A}}(X')|$. Thus, Theorem 3.7 gives $J \cup s_{\mathcal{A}}(F) \in L_{\mathcal{A}}$.

For the last assertion, take $J = s_{\mathcal{A}}(e) - \{h\}$ and $F = \{e\}$. $\square$

The next result gives conditions under which the support of a set is, or is not, closed.

Theorem 3.10. *Let $\mathcal{A} = (A_i : i \in [r])$ and $\mathcal{B} = (B_i : i \in [r])$ be presentations of M.*

(1) *If the presentation $\mathcal{A}$ is maximal, then $s_{\mathcal{A}}(X) \in L_{\mathcal{A}}$ for all $X \subseteq E(M)$.*
(2) *Assume $\mathcal{A} \prec \mathcal{B}$. For $X \subseteq E(M)$, if $s_{\mathcal{A}}(X) \neq s_{\mathcal{B}}(X)$, then $s_{\mathcal{A}}(X) \notin L_{\mathcal{B}}$.*

Proof. We start with an observation. For an element $e \in E(M)$, set $I = s_{\mathcal{A}}(e)$. Since e and x are in the same sets in $\mathcal{A}^I$, the transposition ϕ on $E(M) \cup \{x\}$ that switches e and x is an automorphism of $M[\mathcal{A}^I]$. Thus, ϕ restricted to $E(M)$ is an isomorphism of M onto $M[\mathcal{A}^I]\backslash e$.

For part (1), since $L_{\mathcal{A}}$ is closed under unions, it suffices to treat a singleton set $\{e\}$. Since $[r] \in L_{\mathcal{A}}$, we may assume that $s_{\mathcal{A}}(e) \neq [r]$. Set $I = s_{\mathcal{A}}(e)$ and fix $h \in [r] - I$. By Lemma 2.6, since $\mathcal{A}$ is maximal, e is not a coloop of $M\backslash A_h$, so, by the isomorphism above, x is not a coloop of $M[\mathcal{A}^I]\backslash(A_h \cup \{e\})$. Thus, x is not a coloop of $M[\mathcal{A}^I]\backslash A_h$, so $I \in L_{\mathcal{A}}$.

For part (2), set $J = s_{\mathcal{A}}(X)$, fix $h \in s_{\mathcal{B}}(X) - J$, and pick $e \in X$ with $h \in s_{\mathcal{B}}(e)$. Set $I = s_{\mathcal{A}}(e)$. Since $\mathcal{A} \prec \mathcal{B}$, the element e is a coloop of $M\backslash A_h$ by Lemma 2.6. By the isomorphism above, x is a coloop of $M[\mathcal{A}^I]\backslash(A_h \cup \{e\})$, and thus of $M[\mathcal{B}^J]\backslash(A_h \cup \{e\})$ by Lemma 2.11, and thus of $M[\mathcal{B}^J]\backslash B_h$. Thus, $J \notin L_{\mathcal{B}}$. $\square$

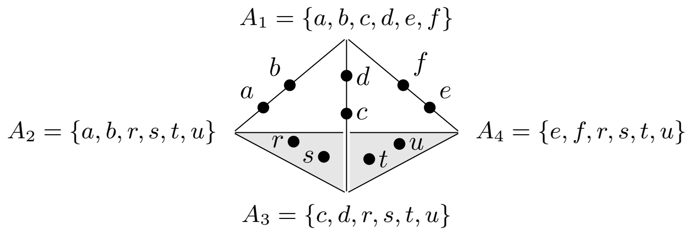

FIGURE 2. A transversal matroid whose minimal presentations are also maximal. The points r, s, t, u are freely in the shaded plane.

Let $\mathcal{A} = (A_i : i \in [r])$ be a maximal presentation of M. Thus, $s_{\mathcal{A}}(e) \in L_{\mathcal{A}}$ for all $e \in E(M)$ by Theorem 3.10. The unions of the sets $s_{\mathcal{A}}(e)$ include the supports of all cyclic flats, but intersections of supports of cyclic flats, which are in $L_{\mathcal{A}}$, need not be intersections of the sets $s_{\mathcal{A}}(e)$, as the example in Figure 2 shows. Each presentation $\mathcal{A}$ of M is both maximal and minimal, so $L_{\mathcal{A}} = 2^{[4]}$. However, $\{2, 3\}$ is not an intersection of the $\mathcal{A}$-supports of singletons. Thus, the sets $s_{\mathcal{A}}(e)$ generate $L_{\mathcal{A}}$, but both their unions and the intersections of such unions are needed to obtain all of $L_{\mathcal{A}}$.

Corollary 3.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be presentations of M with $\mathcal{A} \prec \mathcal{B}$. The sublattice $L_{\mathcal{B}}$ of $L_{\mathcal{A}}$ is a proper sublattice of $L_{\mathcal{A}}$ if either of the conditions below holds.*

(1) *There is an $e \in E(M)$ and $h \in s_{\mathcal{A}}(e)$ with $s_{\mathcal{A}}(e) - \{h\} \in L_{\mathcal{B}}$ and $s_{\mathcal{A}}(e) \neq s_{\mathcal{B}}(e)$.*
(2) *For each $I \in 2^{[r]} - L_{\mathcal{B}}$, there is some $h \in I$ with $I - \{h\} \in L_{\mathcal{B}}$.*

Proof. Condition (1), Corollary 3.9, and Theorem 3.10 give $s_{\mathcal{A}}(e) \in L_{\mathcal{A}} - L_{\mathcal{B}}$. For the second condition, since $\mathcal{A} \prec \mathcal{B}$, there is an $e \in E(M)$ with $s_{\mathcal{A}}(e) \neq s_{\mathcal{B}}(e)$, so condition (1) applies. □

3.4. **The intersection of $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$.** We show that, for presentations $\mathcal{A}$ and $\mathcal{B}$ of a transversal matroid M, the intersection $T_{\mathcal{A}} \cap T_{\mathcal{B}}$ is a sublattice of $T_{\mathcal{A}}$ and of $T_{\mathcal{B}}$, so for pairs of extensions that are in both of these lattices, their meet in $T_{\mathcal{A}}$ is their meet in $T_{\mathcal{B}}$, and likewise for joins. This line of inquiry is motivated in part by the following question [4, Problem 4.1]: is the set of all rank-preserving single-element transversal extensions of a transversal matroid, ordered by the weak order, a lattice? An affirmative answer would provide a transversal counterpart of the following well-known result of Crapo [8]: the set of all single-element extensions of a matroid M, ordered by the weak order, is a lattice. (This lattice is called the lattice of extensions of M.) While it is far from addressing the question about the transversal extensions of a transversal matroid M, the next result, from [4], shows that the join in $T_{\mathcal{A}}$ is the join in the lattice of extensions of M.

Lemma 3.12. *Let $\mathcal{A}$ be a presentation of M, and $r = r(M)$. For any subsets I and J of $[r]$, the join of $M[\mathcal{A}^I]$ and $M[\mathcal{A}^J]$ in the lattice of extensions of M is transversal and is $M[\mathcal{A}^{I \cup J}]$.*

Corollary 3.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be presentations of a transversal matroid M. If M_1 and M_2 are in both $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$, then their join in $T_{\mathcal{A}}$ is their join in $T_{\mathcal{B}}$.*

Proof. Since M_1 and M_2 are in both $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$, there are sets I_1 and I_2 in $L_{\mathcal{A}}$, and sets J_1 and J_2 in $L_{\mathcal{B}}$, with $M[\mathcal{A}^{I_1}] = M[\mathcal{B}^{J_1}] = M_1$ and $M[\mathcal{A}^{I_2}] = M[\mathcal{B}^{J_2}] = M_2$. By the isomorphism in Theorem 3.4, the join of M_1 and M_2 in $T_{\mathcal{A}}$ is $M[\mathcal{A}^{I_1 \cup I_2}]$, and that in $T_{\mathcal{B}}$

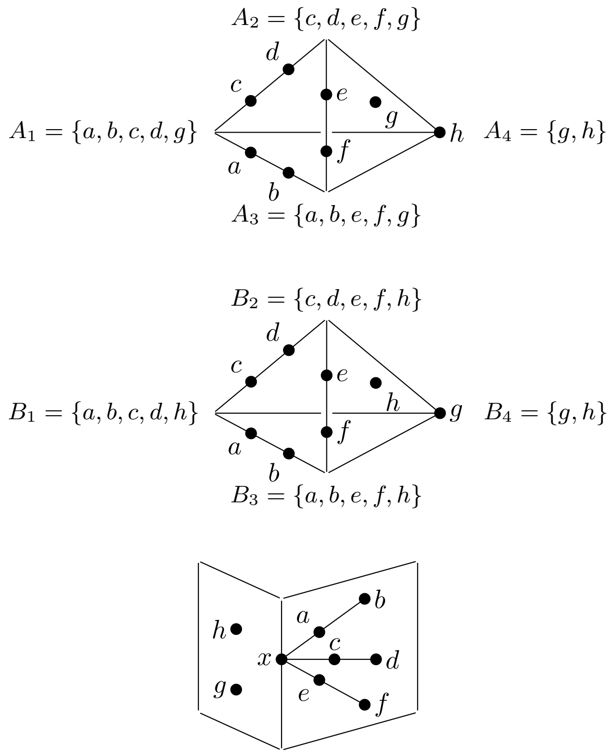

FIGURE 3. The presentations and the meet of the extensions discussed in Example 2. In the first figure, g is in no proper face of the simplex; in the second, h is in no proper face.

is $M[\mathcal{B}^{J_1 \cup J_2}]$. As claimed, these matroids are equal since, by Lemma 3.12,

$$M[\mathcal{A}^{I_1 \cup I_2}] = M[\mathcal{A}^{I_1}] \vee M[\mathcal{A}^{I_2}] = M[\mathcal{B}^{J_1}] \vee M[\mathcal{B}^{J_2}] = M[\mathcal{B}^{J_1 \cup J_2}], \tag{3.2}$$

where $\vee$ denotes the join in the lattice of extensions of M. □

The situation for meets is more complex, as the example below illustrates.

EXAMPLE 2. Consider the matroid M shown in the first two diagrams in Figure 3, and the two presentations given there. In the extension $M_1 = M[\mathcal{A}^{\{1\}}] = M[\mathcal{B}^{\{1\}}]$, both $\{x, a, b\}$ and $\{x, c, d\}$ are lines. In the extension $M_2 = M[\mathcal{A}^{\{2\}}] = M[\mathcal{B}^{\{2\}}]$, both $\{x, c, d\}$ and $\{x, e, f\}$ are lines. In the meet of M_1 and M_2 in the lattice of extensions of M, each of $\{x, a, b\}$, $\{x, c, d\}$ and $\{x, e, f\}$ is dependent; this meet, which is shown in the third diagram in Figure 3, is not transversal. One way to see this is that the three coplanar 3-point lines through x are incompatible with the affine representation described at the end of Section 2. That view also implies that the meet of M_1 and M_2 in both $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ is formed by extending M by a loop.

This example illustrates the next result: the meet of M_1 and M_2 in $T_{\mathcal{A}}$ is their meet in $T_{\mathcal{B}}$ (even though these can differ from their meet in the lattice of all extensions).

Theorem 3.14. *If $\mathcal{A}$ and $\mathcal{B}$ are presentations of M, then the set*

$$L_{\mathcal{A},\mathcal{B}} = \{I \in L_{\mathcal{A}} \,:\, M[\mathcal{A}^I] = M[\mathcal{B}^J] \text{ for some } J \in L_{\mathcal{B}}\}$$

is a sublattice of $L_{\mathcal{A}}$. The sublattices $L_{\mathcal{A},\mathcal{B}}$, of $L_{\mathcal{A}}$, and $L_{\mathcal{B},\mathcal{A}}$, of $L_{\mathcal{B}}$, are isomorphic, and $T_{\mathcal{A}} \cap T_{\mathcal{B}}$ is a sublattice of both $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$.

The proof of this theorem uses the following result from [4].

Lemma 3.15. *Let M be $M[\mathcal{A}]$. For subsets X and Y of $E(M)$, if $r(X) = |s_{\mathcal{A}}(X)|$ and $r(Y) = |s_{\mathcal{A}}(Y)|$, then $r(X \cup Y) = |s_{\mathcal{A}}(X \cup Y)|$.*

Proof of Theorem 3.14. The closure of $L_{\mathcal{A},\mathcal{B}}$ under unions follows from the argument that gives equation (3.2). We next show that the closure of $L_{\mathcal{A},\mathcal{B}}$ under intersections follows from statement (3.14.1), which we then prove.

(3.14.1) *For subsets $X_1, X_2, \ldots, X_t$ of $E(M)$, if $|s_{\mathcal{A}}(X_k)| = r(X_k) = |s_{\mathcal{B}}(X_k)|$ for all $k \in [t]$, then $\bigcap_{k=1}^{t} s_{\mathcal{A}}(X_k) \in L_{\mathcal{A},\mathcal{B}}$.*

To see why proving this statement suffices, consider a pair $I_1 \in L_{\mathcal{A}}$ and $J_1 \in L_{\mathcal{B}}$ with $M[\mathcal{A}^{I_1}] = M[\mathcal{B}^{J_1}]$; let M' denote this extension of M. By equation (3.1),

$$I_1 = \bigcap_{C \in \mathcal{C}_x} s_{\mathcal{A}}(C - \{x\}) \qquad \text{and} \qquad J_1 = \bigcap_{C \in \mathcal{C}_x} s_{\mathcal{B}}(C - \{x\}),$$

where $\mathcal{C}_x$ is the set of circuits of M' that contain x. Now $s_{\mathcal{A}^{I_1}}(C) = s_{\mathcal{A}}(C - \{x\})$ for all $C \in \mathcal{C}_x$, so Lemma 2.4 gives $|s_{\mathcal{A}}(C - \{x\})| = r(C - \{x\}) = |C - \{x\}|$, and the corresponding statements hold for $s_{\mathcal{B}}(C - \{x\})$. The corresponding conclusions also hold for any other pair $I_2 \in L_{\mathcal{A}}$ and $J_2 \in L_{\mathcal{B}}$ with $M[\mathcal{A}^{I_2}] = M[\mathcal{B}^{J_2}]$, so $I_1 \cap I_2$ has the form $\bigcap_{k=1}^{t} s_{\mathcal{A}}(X_k)$ that the claim treats.

The case $t = 1$ merits special attention: if $|s_{\mathcal{A}}(X)| = r(X) = |s_{\mathcal{B}}(X)|$ for some $X \subseteq E(M)$, then $s_{\mathcal{A}}(X) \in L_{\mathcal{A},\mathcal{B}}$ since $M[\mathcal{A}^{s_{\mathcal{A}}(X)}]$ and $M[\mathcal{B}^{s_{\mathcal{B}}(X)}]$ are, by Lemma 2.5, both the principal extension $M +_X x$ of M.

Let the sets $X_1, X_2, \ldots, X_t$ be as in statement (3.14.1). Set $I = \bigcap_{k=1}^{t} s_{\mathcal{A}}(X_k)$ and $J = \bigcap_{k=1}^{t} s_{\mathcal{B}}(X_k)$. To prove the equality $M[\mathcal{A}^I] = M[\mathcal{B}^J]$, which proves statement (3.14.1), by symmetry it suffices to prove that each circuit C of $M[\mathcal{A}^I]$ that contains x is dependent in $M[\mathcal{B}^J]$. Fix such a circuit C of $M[\mathcal{A}^I]$.

We claim that for each $k \in [t]$, we have

$$\big|s_{\mathcal{A}}\big((C - \{x\}) \cup X_k\big)\big| = r\big((C - \{x\}) \cup X_k\big) = \big|s_{\mathcal{B}}\big((C - \{x\}) \cup X_k\big)\big|. \tag{3.3}$$

To see this, let cl be the closure operator of M, and cl_I that of $M[\mathcal{A}^I]$. For any $y \in C - \{x\}$,

$$\mathrm{cl}\big((C - \{x, y\}) \cup X_k\big) = \mathrm{cl}_I\big((C - \{x, y\}) \cup X_k\big) - \{x\}.$$

Lemma 2.4 gives $x \in \mathrm{cl}_I(X_k)$. Thus, y is in $\mathrm{cl}_I\big((C - \{x, y\}) \cup X_k\big)$ since C is a circuit of $M[\mathcal{A}^I]$. Thus, $y \in \mathrm{cl}\big((C - \{x, y\}) \cup X_k\big)$. By the formulation of closure in terms of circuits (as in [12, Proposition 1.4.11]), it follows that each $y \in C - (X_k \cup \{x\})$ is in some circuit, say C_y, of M with $C_y \subseteq X_k \cup (C - \{x\})$. Now $|s_{\mathcal{A}}(C_y)| = r(C_y) = |s_{\mathcal{B}}(C_y)|$ by Lemma 2.4. Since this applies for each $y \in C - (X_k \cup \{x\})$, and since we also have $|s_{\mathcal{A}}(X_k)| = r(X_k) = |s_{\mathcal{B}}(X_k)|$, equation (3.3) now follows from Lemma 3.15.

From equation (3.3), another application of Lemma 3.15 gives

$$\Big|s_{\mathcal{A}}\Big((C-\{x\}) \cup \big(\bigcup_{k \in P} X_k\big)\Big)\Big| = r\Big((C-\{x\}) \cup \big(\bigcup_{k \in P} X_k\big)\Big) = \Big|s_{\mathcal{B}}\Big((C-\{x\}) \cup \big(\bigcup_{k \in P} X_k\big)\Big)\Big|$$

for any non-empty subset P of $[t]$. Thus, for any such P,

$$\Big|\bigcup_{k \in P} s_{\mathcal{A}}\big((C - \{x\}) \cup X_k\big)\Big| = \Big|\bigcup_{k \in P} s_{\mathcal{B}}\big((C - \{x\}) \cup X_k\big)\Big|.$$

Now

$$\begin{aligned}\bigcap_{k=1}^{t} s_{\mathcal{A}}\big((C-\{x\})\cup X_k\big) &= \bigcap_{k=1}^{t}\big(s_{\mathcal{A}}(C-\{x\})\cup s_{\mathcal{A}}(X_k)\big)\\ &= s_{\mathcal{A}}(C-\{x\})\cup\Big(\bigcap_{k=1}^{t} s_{\mathcal{A}}(X_k)\Big)\\ &= s_{\mathcal{A}}(C-\{x\})\cup I\\ &= s_{\mathcal{A}^I}(C).\end{aligned}$$

The same argument applies to $\mathcal{B}$ and gives

$$s_{\mathcal{B}^J}(C) = \bigcap_{k=1}^{t} s_{\mathcal{B}}\big((C-\{x\})\cup X_k\big).$$

The deductions in the previous two paragraphs and inclusion-exclusion give

$$\begin{aligned}|s_{\mathcal{A}^I}(C)| &= \Big|\bigcap_{k=1}^{t} s_{\mathcal{A}}\big((C-\{x\})\cup X_k\big)\Big|\\ &= \sum_{P\subseteq[t]\,:\,P\neq\emptyset}(-1)^{|P|+1}\Big|\bigcup_{k\in P} s_{\mathcal{A}}\big((C-\{x\})\cup X_k\big)\Big|\\ &= \sum_{P\subseteq[t]\,:\,P\neq\emptyset}(-1)^{|P|+1}\Big|\bigcup_{k\in P} s_{\mathcal{B}}\big((C-\{x\})\cup X_k\big)\Big|\\ &= \Big|\bigcap_{k=1}^{t} s_{\mathcal{B}}\big((C-\{x\})\cup X_k\big)\Big|\\ &= |s_{\mathcal{B}^J}(C)|.\end{aligned}$$

Since C is a circuit of $M[\mathcal{A}^I]$, we have $|s_{\mathcal{A}^I}(C)| < |C|$. Thus $|s_{\mathcal{B}^J}(C)| < |C|$, so C is dependent in $M[\mathcal{B}^J]$, as needed.

The assertions about $L_{\mathcal{B},\mathcal{A}}$ and $T_{\mathcal{A}}\cap T_{\mathcal{B}}$ now follow easily. $\square$

The proof of Theorem 3.14 and its reduction to statement (3.14.1) give the following alternative description of $L_{\mathcal{A},\mathcal{B}}$.

Theorem 3.16. *For presentations $\mathcal{A}$ and $\mathcal{B}$ of M, the sublattice $L_{\mathcal{A},\mathcal{B}}$ of $L_{\mathcal{A}}$ consists of the sets $I\in L_{\mathcal{A}}$ that satisfy condition ($*$), as well as all intersections of such sets:*

($*$) $I = s_{\mathcal{A}}(X)$ *for some* $X\subseteq E(M)$ *with* $|s_{\mathcal{A}}(X)| = r(X) = |s_{\mathcal{B}}(X)|$.

The sets I that satisfy condition ($*$) correspond to the principal extensions $M +_X x$ of M that are common to $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$.

We conclude this section with two corollaries. Note that we can iterate the operation of extending set systems to get $(\mathcal{A}^{I_1})^{I_2}$, where x_1 is added in $\mathcal{A}^{I_1}$, and x_2 is added in $(\mathcal{A}^{I_1})^{I_2}$. We next show that such extensions, using sets in $L_{\mathcal{A},\mathcal{B}}$, are compatible.

Corollary 3.17. *If $M[\mathcal{A}^{I_1}] = M[\mathcal{B}^{J_1}]$ and $M[\mathcal{A}^{I_2}] = M[\mathcal{B}^{J_2}]$ for some sets $I_1, I_2\in L_{\mathcal{A}}$ and $J_1, J_2\in L_{\mathcal{B}}$, then $M[(\mathcal{A}^{I_1})^{I_2}] = M[(\mathcal{B}^{J_1})^{J_2}]$.*

Proof. The result follows from two observations: (i) Theorem 3.7 yields $I_2 \in L_{\mathcal{A}^{I_1}}$ and $J_2 \in L_{\mathcal{B}^{J_1}}$; (ii) if I_2 and X satisfy condition $(*)$ above in M, then so do I_2 and X in $M[\mathcal{A}^{I_1}]$, and likewise for intersections of sets that satisfy condition $(*)$. $\square$

Corollary 3.18. *For $I \in L_{\mathcal{A}}$ and $J \in L_{\mathcal{B}}$, if $M[\mathcal{A}^I] = M[\mathcal{B}^J]$, then $|I| = |J|$.*

Proof. Apply Corollary 3.17 repeatedly, with each $I_h = I$ and each $J_h = J$, until the set of added elements is cyclic in the extension; the rank of this cyclic set must be both $|I|$ and $|J|$. $\square$

3.5. **How to get any finite distributive lattice.** We show that each sublattice of $2^{[r]}$ that includes both $\emptyset$ and $[r]$ is the lattice $L_{\mathcal{A}}$ for some presentation $\mathcal{A}$ of some transversal matroid of rank r; indeed, we prove two refinements of this result. Up to isomorphism, this result covers all finite distributive lattices since each such lattice L is isomorphic to the lattice of order ideals of some finite ordered set (specifically, the induced order on the set of join-irreducible elements of L; see, e.g., [1, Theorem II.2.5]). Combining the result below with Theorem 3.4 shows any distributive lattice is isomorphic to $T_{\mathcal{A}}$ for some presentation $\mathcal{A}$ of some transversal matroid.

Theorem 3.19. *Let L be a sublattice of $2^{[r]}$ that contains both $\emptyset$ and $[r]$.*

(1) *There is a rank-r transversal matroid M and maximal presentation $\mathcal{A}$ of M with $L = L_{\mathcal{A}}$.*
(2) *For any $n \geq r$, there is a presentation $\mathcal{B}$ of the uniform matroid $U_{r,n}$ with $L = L_{\mathcal{B}}$.*

Proof. To prove assertion (1), for each non-empty set $I \in L$, let X_I be a set of $|I| + 1$ elements that is disjoint from all other such sets X_J. For i with $1 \leq i \leq r$, let

$$A_i = \bigcup_{I \in L \,:\, i \in I} X_I,$$

so the elements of X_I are in exactly $|I|$ of the sets A_i (counting multiplicity; we may have $A_i = A_j$ even if $i \neq j$). Let $\mathcal{A} = (A_i \,:\, i \in [r])$ and let M be the matroid $M[\mathcal{A}]$ on

$$E(M) = \bigcup_{I \in L \,:\, I \neq \emptyset} X_I = \bigcup_{i=1}^{r} A_i.$$

Thus, if $e \in X_I$, then $s_{\mathcal{A}}(e) = I$. The presentation $\mathcal{A}$ of M is maximal since, with $|X_I| > |I|$ and $s_{\mathcal{A}}(X_I) = I$, the set X_I is dependent in M, yet if we adjoin any element of X_I to any set A_j with $j \notin I$, then the resulting set system $\mathcal{A}'$ has a matching of X_I, so X_I is independent in $M[\mathcal{A}']$. It now follows from Theorem 3.10 that $L \subseteq L_{\mathcal{A}}$. Since L and $L_{\mathcal{A}}$ are sublattices of $2^{[r]}$ and $s_{\mathcal{A}}(e) \in L$ for all $e \in E(M)$ by construction, we get $s_{\mathcal{A}}(F) \in L$ for each cyclic flat F of M, so Corollary 3.8 gives $L_{\mathcal{A}} \subseteq L$. Thus, $L_{\mathcal{A}} = L$.

Figure 4 illustrates the proof of assertion (2). Let $[n]$ be the ground set of $U_{r,n}$. For $I \in L$, let I_0 be the (possibly empty) set of elements that occur first in I, that is,

$$I_0 = I - \bigcup_{J \in L \,:\, J \subsetneq I} J.$$

Since L is closed under intersection, for each $i \in [r]$, there is exactly one $I \in L$ with $i \in I_0$; using that I, set

$$B_i = ([n] - [r]) \cup \bigcup_{J \in L \,:\, I \subseteq J} J_0.$$

By construction, $|\mathcal{B}| = r$ and $i \in B_i$, so $[r]$ is a basis of $M[\mathcal{B}]$. Since $[n] - [r] \subseteq B_i$ for all $i \in [r]$, it follows that $M[\mathcal{B}]$ is the uniform matroid $U_{r,n}$. For $i \in I_0$ and $j \in J_0$, we

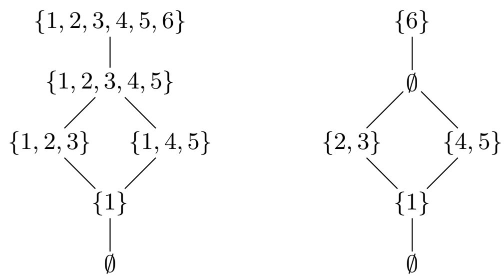

FIGURE 4. An example, for $U_{6,7}$, of the construction of $\mathcal{B}$ in the proof of Theorem 3.19, with L on the left and the sets I_0 on the right. The presentation has $B_1 = \{1,2,3,4,5,6,7\}$, $B_2 = B_3 = \{2,3,6,7\}$, $B_4 = B_5 = \{4,5,6,7\}$, and $B_6 = \{6,7\}$.

have $i \in B_j$ if and only if $J \subseteq I$, so $s_{\mathcal{B}}(i) = I$. Since L is closed under unions, we get $s_{\mathcal{B}}(X) \in L$ for all $X \subseteq [r]$. Also, each set $I \in L$ is independent in $U_{r,n}$ and $s_{\mathcal{B}}(I) = I$. From these observations and Theorem 3.7, we get $L = L_{\mathcal{B}}$. □

3.6. **Irreducible elements.** An element a in a lattice L is *join-irreducible* if (i) a is not the least element of L and (ii) if $a = b \vee c$, then $a \in \{b, c\}$. Dually, a is *meet-irreducible* if (i′) a is not the greatest element of L and (ii′) if $a = b \wedge c$, then $a \in \{b, c\}$. (While not all authors include them, conditions (i) and (i′) shorten the wording of results.)

The irreducible elements of a finite distributive lattice L are of great interest. The order induced on the set of join-irreducibles of L is isomorphic to that induced on its set of meet-irreducibles, and the lattice of order ideals of each of these induced suborders of L is isomorphic to L itself. (See, e.g., [1, Theorem II.2.5 and Corollary II.2.7].) Thus, the rank of L is the number of join-irreducibles in L, which is also its number of meet-irreducibles.

We now study the irreducible elements of the lattices $L_{\mathcal{A}}$ introduced above.

The least set S_i in $L_{\mathcal{A}}$ that contains a given element $i \in [r]$ is $\bigcap_{J \in L_{\mathcal{A}} \,:\, i \in J} J$. The sets S_i are not limited to the atoms of $L_{\mathcal{A}}$; see the examples in Figure 1. Clearly S_i is join-irreducible. Each set U in $L_{\mathcal{A}}$ is $\bigcup_{i \in U} S_i$, so there are no other join-irreducibles of $L_{\mathcal{A}}$. Thus, the number of join-irreducibles is the number of distinct sets S_i. Note that if A_i and A_j in $\mathcal{A}$ are equal, then $S_i = S_j$ since, for $X \subseteq E(M)$, we have $i \in s_{\mathcal{A}}(X)$ if and only if $j \in s_{\mathcal{A}}(X)$. Thus, the number of join-irreducible sets in $L_{\mathcal{A}}$ is at most the number of distinct sets in $\mathcal{A}$. As Example 1 shows, this bound can be strict (there, $\mathcal{A}$ has three distinct sets but $L_{\mathcal{A}}$ has only one join-irreducible; likewise for $\mathcal{B}$).

The greatest set in $L_{\mathcal{A}}$ that does not contain a given element $i \in [r]$ is $\bigcup_{J \in L_{\mathcal{A}} \,:\, i \notin J} J$. An argument like that above, or an application of order-duality, shows that these are the meet-irreducibles of $L_{\mathcal{A}}$. By the remark after the proof of Theorem 3.7, each meet-irreducible element of $L_{\mathcal{A}}$ corresponds to a principal extension of M; the converse is false, since for instance, in either example in Figure 1, the set $\{2, 3\}$ corresponds to a principal extension, but $\{2, 3\}$ is the meet of the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$ in $L_{\mathcal{A}}$.

We now identify a join-sublattice $L'_{\mathcal{A}}$ of $L_{\mathcal{A}}$ that, by Theorem 3.7, has the same the meet-irreducibles, thereby reducing the problem of finding the meet-irreducibles of $L_{\mathcal{A}}$ to the same problem on a potentially smaller lattice. Set

$$L'_{\mathcal{A}} = \{s_{\mathcal{A}}(X) \,:\, X \subseteq E(M),\, |s_{\mathcal{A}}(X)| = r(X)\}.$$

(Adding the condition that X is independent would not change $L'_{\mathcal{A}}$.) By Theorem 3.7, $L'_{\mathcal{A}} \subseteq L_{\mathcal{A}}$ and $L'_{\mathcal{A}}$ generates $L_{\mathcal{A}}$ since $L_{\mathcal{A}}$ consists precisely of the intersections of the sets in $L'_{\mathcal{A}}$. Lemma 3.15 shows that $L'_{\mathcal{A}}$ is a join-sublattice of $L_{\mathcal{A}}$.

Each lattice is isomorphic to $L'_{\mathcal{A}}$ for a maximal presentation $\mathcal{A}$ of some transversal matroid (see the proof of [3, Theorem 2.1]). By Corollary 3.8, when the presentation $\mathcal{A}$ is maximal, the same conclusions hold for the (often smaller) lattice

$$L''_{\mathcal{A}} = \{s_{\mathcal{A}}(X) \,:\, X \text{ is a cyclic flat of } M\} \cup [r].$$

4. Applications

Theorems 4.1 and 4.5 below are applications of the results in Section 3. Both results stem from the observation that proper sublattices of $2^{[r]}$ must be substantially smaller than $2^{[r]}$. (The special case of maximal proper sublattices of $2^{[r]}$ have been studied in other settings, such as finite topologies; see, e.g., Sharp [14] and Stephen [15].)

Theorem 4.1. *Let M be a transversal matroid of rank r, and let $\mathcal{A}^i$ be a presentation of M that has rank i in the ordered set of presentations of M. If $1 \leq i < r$, then*

$$|T_{\mathcal{A}^i}| = |L_{\mathcal{A}^i}| \leq \left(\frac{1}{2} + \frac{1}{2^{i+1}}\right)2^r;$$

these bounds are sharp. Also, if $i \geq r$, then $|T_{\mathcal{A}^i}| = |L_{\mathcal{A}^i}| \leq 2^{r-1}$.

We first give examples to show that, for $1 \leq i < r$, the bounds are sharp. (These examples, which play a role in the proof of the bound, have coloops; to get examples without coloops, take free extensions of these.) Let $\mathcal{B} = (B_2, B_3, \ldots, B_r)$ be a minimal presentation of a transversal matroid N of rank $r-1$. Fix an element $e \notin E(M)$ and let M be the direct sum of N and the rank-1 matroid on $\{e\}$. For $0 \leq k < r$, define $\mathcal{A}^k = (A^k_i \,:\, i \in [r])$ by

$$A^k_i = \begin{cases} \{e\}, & \text{if } i = 1, \\ B_i \cup \{e\}, & \text{if } 2 \leq i \leq k+1, \\ B_i, & \text{otherwise.} \end{cases}$$

Thus, $s_{\mathcal{A}^k}(e) = [k+1]$. Each $\mathcal{A}^k$ is a presentation of M, the presentation $\mathcal{A}^0$ is minimal, and $\mathcal{A}^{k-1} \prec\!\cdot\, \mathcal{A}^k$ for $k \geq 1$. Thus, $\mathcal{A}^k$ has rank k in the ordered set of presentations. Since $\mathcal{B}$ is a minimal presentation of N, each subset of $\{2, 3, \ldots, r\}$ is in $L_{\mathcal{A}^k}$. Thus, since $s_{\mathcal{A}^k}(e) = [k+1]$, Corollary 3.9 implies that all supersets of $[k+1]$ are in $L_{\mathcal{A}^k}$. Since $1 \in s_{\mathcal{A}^k}(X)$ if and only if $e \in X$, by Theorem 3.7 the sets in $L_{\mathcal{A}^k}$ that contain 1 must contain all of $[k+1]$. Thus, $L_{\mathcal{A}^k}$ consists of the subsets of $[r]$ that either do not contain 1 or contain all of $[k+1]$. For reasons that Lemma 4.3 will reveal, it is useful to recast this as follows: $L_{\mathcal{A}^k}$ is the complement, in $2^{[r]}$, of the union of the intervals

$$[\{1\}, \overline{\{2\}}],\ \ [\{1,2\}, \overline{\{3\}}],\ \ [\{1,2,3\}, \overline{\{4\}}],\ \ \ldots,\ \ [\{1,2,\ldots,k\}, \overline{\{k+1\}}],$$

where $\overline{X}$ denotes the complement of the set X. From the first description of $L_{\mathcal{A}^k}$, we get

$$|L_{\mathcal{A}^k}| = 2^{r-1} + 2^{r-(k+1)} = \left(\frac{1}{2} + \frac{1}{2^{k+1}}\right)2^r.$$

The proof of the bound in Theorem 4.1 uses Lemma 4.3, which catalogs the sublattices of $2^{[r]}$ that have more than 2^{r-1} elements. The proof of that lemma uses the following result by Chen, Koh, and Tan [7] (see the proof in Rival [13]).

Lemma 4.2. *Let $\mathcal{J}$ be the set of join-irreducibles of a finite distributive lattice L, and $\mathcal{M}$ its set of meet-irreducibles. The maximal proper sublattices of L are precisely the differences $L - [a,b]$ where the interval $[a,b]$ in L satisfies $[a,b] \cap \mathcal{J} = \{a\}$ and $[a,b] \cap \mathcal{M} = \{b\}$.*

Lemma 4.3. *Up to permutations of $[r]$, the sublattices of $2^{[r]}$ that have more than 2^{r-1} elements are $L_i = 2^{[r]} - U_i$ and $L'_i = 2^{[r]} - U'_i$, for $1 \leq i < r$, where*

$$U_i = \bigcup_{j\,:\,1\leq j\leq i} [\{1,2,\ldots,j\}, \overline{\{j+1\}}] \quad \textit{and} \quad U'_i = \bigcup_{j\,:\,1\leq j\leq i} [\{j+1\}, \overline{\{1,2,\ldots,j\}}],$$

and $L_V = 2^{[r]} - V$ where $V = [\{1\}, \overline{\{2\}}] \cup [\{3\}, \overline{\{4\}}]$. Thus, $|L_i| = |L'_i| = \left(\frac{1}{2} + \frac{1}{2^{i+1}}\right)2^r$ and $|L_V| = \frac{9}{16} \cdot 2^r$. Also, L_V is not contained in any sublattice L of $2^{[r]}$ with $|L| = \frac{5}{8} \cdot 2^r$.

Proof. To prove this result, we apply Lemma 4.2 recursively. To simplify the argument, note that U'_i is the image of U_i under the complementation map $X \mapsto \overline{X}$ (which is order-reversing) of $2^{[r]}$; this allows us to pursue only the lattices L_V and $L_1, L_2, \ldots, L_{r-1}$ below.

The join-irreducibles of $2^{[r]}$ are the singleton sets, and the meet-irreducibles are their complements, so by Lemma 4.2, the maximal proper sublattices of $2^{[r]}$ are L_1 and its images under permutations of $[r]$ (the lattice L'_1 is obtained by such a permutation).

To verify the assertions below about join-irreducibles, note that (i) each join-irreducible of L_{i-1} that is also in L_i is join-irreducible in L_i, and (ii) L_i has at most r join-irreducibles. (The second statement holds since the rank of a distributive lattice is its number of join-irreducibles; see [1, Corollary II.2.11].) Similar observations apply to meet-irreducibles.

We now find the maximal proper sublattices of $L_1 = 2^{[r]} - [\{1\}, \overline{\{2\}}]$. Its join-irreducibles are $\{i\}$, for $2 \leq i \leq r$, along with $\{1,2\}$; its meet-irreducibles are $\overline{\{i\}}$, for $i \in [r] - \{2\}$, along with $\overline{\{1,2\}}$. Up to the map $X \mapsto \overline{X}$ (which maps L_2 to L'_2) and permuting $3, 4, \ldots, r$, there are three maximal proper sublattices, namely

(1) $L_2 = L_1 - [\{1,2\}, \overline{\{3\}}]$, which has $\frac{5}{8} \cdot 2^r$ elements,
(2) $L_V = L_1 - [\{3\}, \overline{\{4\}}]$, which has $\frac{9}{16} \cdot 2^r$ elements, and
(3) $L_1 - [\{2\}, \overline{\{1\}}]$, which has 2^{r-1} elements.

(The join-irreducible $\{1,2\}$ is in $[\{2\}, \overline{\{3\}}]$, so this interval is not listed. Likewise for $\overline{\{1,2\}}$ and $[\{3\}, \overline{\{1\}}]$.) Only L_2 and L_V are of interest for the lemma.

The join-irreducibles of L_V are $\{i\}$, for $i \in [r] - \{1,3\}$, along with $\{1,2\}$ and $\{3,4\}$; its meet-irreducibles are $\overline{\{j\}}$, for $j \in [r] - \{2,4\}$, along with $\overline{\{1,2\}}$ and $\overline{\{3,4\}}$. Up to switching the pair $(1,2)$ with the pair $(3,4)$, permuting $5, 6, \ldots, r$, and the map $X \mapsto \overline{X}$, there are three maximal proper sublattices of L_V (omitting the case covered by (3) above):

(4) $L_V - [\{1,2\}, \overline{\{3,4\}}]$, which has 2^{r-1} elements,
(5) $L_V - [\{1,2\}, \overline{\{5\}}]$, which has $\frac{15}{32} \cdot 2^r$ elements, and
(6) $L_V - [\{5\}, \overline{\{6\}}]$, which has $\frac{27}{64} \cdot 2^r$ elements.

Thus, no proper sublattices of L_V have more than 2^{r-1} elements.

To complete the proof, we induct to show that for i with $3 \leq i < r$, the only maximal proper sublattice L of L_{i-1} with $|L| > 2^{r-1}$ is L_i, up to permuting elements. We include the following conditions in the induction argument (see Figure 5):

(i) the join-irreducibles of L_{i-1} are $\{j\}$, for $1 < j \leq r$, along with $[i]$, and
(ii) the meet-irreducibles of L_{i-1} are $\overline{\{1\}}$ and $\overline{\{k\}}$, for $i < k \leq r$, along with $\overline{\{1,t\}}$ where $2 \leq t \leq i$.

Conditions (i) and (ii) are easy to see in the base case, $i = 3$. We use the same argument for the base case as for the inductive step. Let L be a maximal proper sublattice of L_{i-1}.

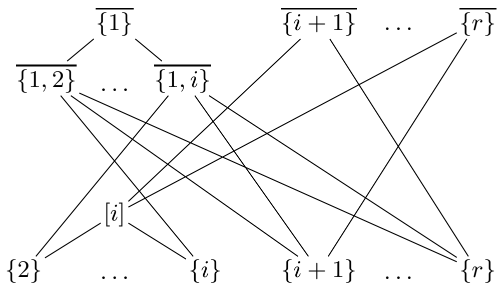

FIGURE 5. The induced order on the irreducibles of L_{i-1}.

If $L = L_{i-1} - [A, B]$ where $|A| = 1$ and $B = \overline{\{1,t\}}$ with $2 \leq t \leq i$, then $[A, B]$ is disjoint from U_{i-1} and has 2^{r-3} elements, so $|L| \leq 2^{r-1}$. If $L = L_{i-1} - [\{j\}, \overline{\{k\}}]$, with j and k distinct elements of $\{i+1, i+2, \dots, r\}$, then $|L| \leq \frac{15}{32} \cdot 2^r$ by case (5) (with relabelling). Thus, up to relabelling, only $L_i = L_{i-1} - [\{1, 2, \dots, i\}, \overline{\{i+1\}}]$ has more than 2^{r-1} elements: $|L_i| = \big(\frac{1}{2} + \frac{1}{2^{i+1}}\big)2^r$. It is easy to check that conditions (i) and (ii) hold for L_i, which completes the induction. □

The last background item we need before proving the upper bounds in Theorem 4.1 is the following lemma from [4].

Lemma 4.4. *Let $\mathcal{A}$ be a presentation of M. Fix $Y \subseteq E(M)$. If $r(M\backslash Y) = r(M)$, then M has a minimal presentation $\mathcal{C}$ with $\mathcal{C} \preceq \mathcal{A}$ so that $s_{\mathcal{C}}(e) = s_{\mathcal{A}}(e)$ for all $e \in Y$.*

Proof of Theorem 4.1. Consider presentations $\mathcal{A}^0 \prec\!\cdot\, \mathcal{A}^1 \prec\!\cdot\, \cdots \prec\!\cdot\, \mathcal{A}^r$ of M where $\mathcal{A}^0$ is minimal. Thus, $\mathcal{A}^j$ has rank j in the order on presentations, and $L_{\mathcal{A}^j}$ is a sublattice of $L_{\mathcal{A}^{j-1}}$. By Lemma 4.3, if $|L_{\mathcal{A}^j}| > 2^{r-1}$, then $|L_{\mathcal{A}^j}| = \big(\frac{1}{2} + \frac{1}{2^{i+1}}\big)2^r$ for some i with $1 \leq i < r$, so it suffices to prove the following statement:

$$\textit{if } |L_{\mathcal{A}^j}| = \Big(\frac{1}{2} + \frac{1}{2^{i+1}}\Big)2^r, \textit{ then } j \leq i.$$

For $i = 1$, assume $|L_{\mathcal{A}^j}| = \frac{3}{4} \cdot 2^r$. By Lemma 4.3, up to permuting $[r]$, we have $L_{\mathcal{A}^j} = 2^{[r]} - [\{1\}, \overline{\{2\}}]$. Condition (2) of Corollary 3.11 holds (h is 1), so $L_{\mathcal{A}^j}$ is properly contained in $L_{\mathcal{A}^{j-1}}$; since $L_{\mathcal{A}^j}$ is a proper sublattice only of $2^{[r]}$, we have $L_{\mathcal{A}^{j-1}} = 2^{[r]}$. Thus, $\mathcal{A}^{j-1}$ is a minimal presentation by Theorem 3.6, so $j - 1 = 0$, so $j = 1$.

For $i = 2$, if $|L_{\mathcal{A}^j}| = \frac{5}{8} \cdot 2^r$, then, by Lemma 4.3, up to permuting $[r]$, the lattice $L_{\mathcal{A}^j}$ is either

$$2^{[r]} - \big([\{1\}, \overline{\{2\}}] \cup [\{1,2\}, \overline{\{3\}}]\big) \qquad \text{or} \qquad 2^{[r]} - \big([\{2\}, \overline{\{1\}}] \cup [\{3\}, \overline{\{1,2\}}]\big).$$

Condition (2) of Corollary 3.11 holds (h is 1 in the first case and either 2 or 3 in the second), so $L_{\mathcal{A}^j}$ is properly contained in $L_{\mathcal{A}^{j-1}}$. Thus, $|L_{\mathcal{A}^{j-1}}| \geq \frac{3}{4} \cdot 2^r$. The previous case gives $j - 1 \leq 1$, so $j \leq 2$.

The general case with $L_{\mathcal{A}^j} = L_i$ or $L_{\mathcal{A}^j} = L'_i$ follows inductively in the same manner. We turn to the only case that requires a more involved argument, namely

$$L_{\mathcal{A}^j} = L_V = 2^{[r]} - \big(\,[\{1\}, \overline{\{2\}}] \cup [\{3\}, \overline{\{4\}}]\,\big).$$

Since $\mathcal{A}^{j-1} \prec\!\cdot\, \mathcal{A}^j$, we have $s_{\mathcal{A}^{j-1}}(e) \subsetneq s_{\mathcal{A}^j}(e)$ for some $e \in E(M)$, so $s_{\mathcal{A}^{j-1}}(e) \notin L_V$ by Theorem 3.10. Thus, $s_{\mathcal{A}^{j-1}}(e) \in [\{1\}, \overline{\{2\}}] \cup [\{3\}, \overline{\{4\}}]$. If $s_{\mathcal{A}^{j-1}}(e)$ is in only one of $[\{1\}, \overline{\{2\}}]$ and $[\{3\}, \overline{\{4\}}]$, then $L_{\mathcal{A}^j}$ is a proper sublattice of $L_{\mathcal{A}^{j-1}}$ by condition (1) of

Corollary 3.11; thus, $|L_{\mathcal{A}^{j-1}}| \geq \frac{3}{4} \cdot 2^r$, so $j-1 \leq 1$, so $j < 3$. We may now assume that $L_{\mathcal{A}^j} = L_{\mathcal{A}^{j-1}}$ and that $s_{\mathcal{A}^{j-1}}(e) \in [\{1\}, \overline{\{2\}}] \cap [\{3\}, \overline{\{4\}}]$.

First assume that for all options for the terms $\mathcal{A}^0, \mathcal{A}^1, \dots, \mathcal{A}^{j-1}$, the only element d with $s_{\mathcal{A}^j}(d) \neq s_{\mathcal{A}^k}(d)$ for some $k < j$ is $d = e$. Lemma 4.4 then implies that e is a coloop of M; also, the presentation of $M\backslash e$ that is obtained by removing e from all sets in $\mathcal{A}^0$ is minimal. This case is covered by the example that we used to show that the bound is sharp, so we may now assume that e is not a coloop of M.

In this case, by Lemma 4.4 with $J = \{e\}$, we can choose $\mathcal{A}^0, \mathcal{A}^1, \dots, \mathcal{A}^{j-2}$ so that $s_{\mathcal{A}^{j-1}}(e) = s_{\mathcal{A}^{j-2}}(e)$. Since $\mathcal{A}^{j-2} \prec\!\cdot\, \mathcal{A}^{j-1}$, we have $s_{\mathcal{A}^{j-2}}(e') \subsetneq s_{\mathcal{A}^{j-1}}(e')$ for some $e' \in E(M)$. Thus, $e' \neq e$. Now $s_{\mathcal{A}^{j-2}}(e') \notin L_V$ by Theorem 3.10, so $s_{\mathcal{A}^{j-2}}(e')$ is in either $[\{1\}, \overline{\{2\}}]$ or $[\{3\}, \overline{\{4\}}]$. If $s_{\mathcal{A}^{j-2}}(e')$ is not in both intervals, then the argument above gives the result, so assume $s_{\mathcal{A}^{j-2}}(e') \in [\{1\}, \overline{\{2\}}] \cap [\{3\}, \overline{\{4\}}]$. Set $F = \{e, e'\}$. Thus,

$$s_{\mathcal{A}^{j-2}}(F) = s_{\mathcal{A}^{j-2}}(e) \cup s_{\mathcal{A}^{j-2}}(e') \in [\{1\}, \overline{\{2\}}] \cap [\{3\}, \overline{\{4\}}].$$

Corollary 3.9 with $J = s_{\mathcal{A}^{j-2}}(F) - \{1, 3\}$, and so $H = \{1, 3\}$, gives $s_{\mathcal{A}^{j-2}}(F) \in L_{\mathcal{A}^{j-2}}$, so $L_{\mathcal{A}^j}$ is a proper sublattice of $L_{\mathcal{A}^{j-2}}$. Lemma 4.3 gives $|L_{\mathcal{A}^{j-2}}| \geq \frac{3}{4} \cdot 2^r$; thus, $j-2 \leq 1$, so $j \leq 3$, as needed. □

Let $\mathcal{A}$ and $\mathcal{B}$ be presentations of M. In Theorem 3.14 we showed that $T_{\mathcal{A}} \cap T_{\mathcal{B}}$ is a sublattice of both $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$. The smallest that $|T_{\mathcal{A}} \cap T_{\mathcal{B}}|$ can be is two, with these two common extensions being the free extension and the extension by a loop; for instance, the two minimal presentations

$$\mathcal{A} = (\{i\} \cup ([2r] - [r]) \,:\, i \in [r]) \quad \text{and} \quad \mathcal{B} = ([r] \cup \{i\} \,:\, i \in [2r] - [r])$$

of $U_{r,2r}$ on $[2r]$ have this property. We conclude with a sharp upper bound on $|T_{\mathcal{A}} \cap T_{\mathcal{B}}|$.

Theorem 4.5. *If the presentations $\mathcal{A} = (A_i \,:\, i \in [r])$ and $\mathcal{B} = (B_i \,:\, i \in [r])$ of M differ by more than just reindexing the sets, then $|T_{\mathcal{A}} \cap T_{\mathcal{B}}| \leq \frac{3}{4} \cdot 2^r$. This bound is sharp.*

Proof. The inequality follows from Theorems 4.1 and 3.14 if either $\mathcal{A}$ or $\mathcal{B}$ is not minimal, so we may assume that both are minimal. As shown in Section 3.2, when $\mathcal{A}$ is minimal, we can reconstruct the sets in $\mathcal{A}$ from $T_{\mathcal{A}}$; thus, by our assumption, $T_{\mathcal{A}} \neq T_{\mathcal{B}}$, so $L_{\mathcal{A},\mathcal{B}}$ is a proper sublattice of $L_{\mathcal{A}}$. Thus, we get the bound by our work above.

To see that this bound is tight, let M be $U_{r-2,r-2} \oplus U_{2,3}$, with $U_{r-2,r-2}$ and $U_{2,3}$ on the sets $\{e_1, e_2, \dots, e_{r-2}\}$ and $\{e_{r-1}, a, b\}$, respectively. Consider the presentations $\mathcal{A} = (A_i \,:\, i \in [r])$ and $\mathcal{B} = (B_i \,:\, i \in [r])$ where $A_i = B_i = \{e_i\}$ for $i \in [r-2]$ and

$$A_{r-1} = \{e_{r-1}, a\}, \qquad B_{r-1} = \{e_{r-1}, b\}, \qquad A_r = B_r = \{a, b\}.$$

By Lemma 2.5, if $I \subseteq [r-1]$, then both $M[\mathcal{A}^I]$ and $M[\mathcal{B}^I]$ are the principal extension $M +_Y x$ where $Y = \{e_i \,:\, i \in I\}$; also, if $\{r-1, r\} \subseteq I \subseteq [r]$, then $M[\mathcal{A}^I]$ and $M[\mathcal{B}^I]$ are both $M +_Y x$ where $Y = \big\{e_i \,:\, i \in I - \{r\}\big\} \cup \{a, b\}$. There are $2^{r-1} + 2^{r-2} = \frac{3}{4} \cdot 2^r$ such sets I, so the bound is optimal. □

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REFERENCES

[1] M. Aigner, *Combinatorial Theory*, (Springer-Verlag, Berlin, New York, 1979).

[2] J. A. Bondy and D. J. A. Welsh, Some results on transversal matroids and constructions for identically self-dual matroids, *Quart. J. Math. Oxford Ser.* **22** (1971) 435–451.

[3] J. Bonin and A. de Mier, The lattice of cyclic flats of a matroid, *Ann. Comb.*, **12** (2008) 155–170.

[4] J. Bonin and A. de Mier, Extensions and presentations of transversal matroids, *European J. Combin.* **50** (2015) 18–29.

[5] R. Brualdi, Transversal matroids, in: *Combinatorial geometries, Encyclopedia Math. Appl., 29*, Cambridge Univ. Press, Cambridge, 1987, 72-97.

[6] R. Brualdi and G. Dinolt, Characterizations of transversal matroids and their presentations, *J. Combin. Theory Ser. B* **12** (1972) 268–286.

[7] C. Chen, K. Koh, and S. Tan, Frattini sublattices of distributive lattices, *Algebra Universalis* **3** (1973) 294–303.

[8] H. H. Crapo, Single-element extensions of matroids, *J. Res. Natl. Bureau Standards Sect. B* **69** (1965) 55–65.

[9] J. Edmonds and D. R. Fulkerson, Transversals and matroid partition, *J. Res. Nat. Bur. Standards Sect. B* **69B** (1965) 147–153.

[10] M. Las Vergnas, Sur les systèmes de représentants distincts d'une famille d'ensembles, *C. R. Acad. Sci. Paris Sér. A-B* **270** (1970) A501–A503.

[11] J. Mason, *Representations of Independence Spaces*, (Ph.D. Dissertation, University of Wisconsin, Madison WI, 1969).

[12] J. G. Oxley, *Matroid Theory*, second edition (Oxford University Press, Oxford, 2011).

[13] I. Rival, Maximal sublattices of finite distributive lattices, *Proc. Amer. Math. Soc.* **37** (1973) 417–420.

[14] H. Sharp, Cardinality of finite topologies, *J. Combinatorial Theory* **5** (1968) 82–86.

[15] D. Stephen, Topology on finite sets, *Amer. Math. Monthly* **75** (1968) 739–741.

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